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Casimir amplitudes in a quantum spherical model with long-range interaction

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Abstract. A d-dimensional quantum model system confined to a general hypercubical geometry with linear spatial size L and "temporal size" $1/T$ (T - temperature of the system) is considered in the spherical approximation under periodic boundary conditions. For a film geometry in different space dimensions $\frac{1}{2}\sigma < d < \frac{3}{2}\sigma$, where $0 < \sigma \leq 2$ is a parameter controlling the decay of the long-range interaction, the free energy and the Casimir amplitudes are given. We have proven that, if $d = \sigma$, the Casimir amplitude of the model, characterizing the leading temperature corrections to its ground state, is $\Delta = -16\zeta(3)/[5\sigma(4\pi)^{\sigma/2}\Gamma(\sigma/2)]$. The last implies that the universal constant $\tilde{c} = 4/5$ of the model remains the same for both short, as well as long-range interactions, if one takes the normalization factor for the Gaussian model to be such that $\tilde{c} = 1$. This is a generalization to the case of long-range interaction of the well known result due to Sachdev. That constant differs from the corresponding one characterizing the leading finite-size corrections at zero temperature which for $d = \sigma = 1$ is $\tilde{c} = 0.606$.

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1 Introduction

The confinement of quantum mechanical vacuum fluctuations of the electromagnetic field causes long-ranged forces between two conducting uncharged plates which is known as (quantum mechanical) Casimir effect [1,2]. The confinement of *critical fluctuations* of an order parameter field induces long-ranged forces between the surfaces of the film [3,4]. This is known as "statistical-mechanical Casimir effect". The Casimir force in statistical-mechanical systems is characterized by the excess free energy due to the *finite-size contributions* to the free energy of the system. In the case of a film geometry $L\times\infty^2$, and under given boundary conditions τ imposed across the direction L, the Casimir force is defined as

$$F^{\tau}_{\mathrm{Casimir}}(T,L) = -\frac{\partial f^{\mathrm{ex}}_{\tau}(T,L)}{\partial L}, \tag{1.1}$$

where $f^{\mathrm{ex}}_{\tau}(T,L)$ is the excess free energy

$$f^{\mathrm{ex}}_{\tau}(T,L) = f_{\tau}(T,L) - Lf_{\mathrm{bulk}}(T). \tag{1.2}$$

Here $f_{\tau}(T,L)$ is the full free energy per unit area and per k_BT, and $f_{\mathrm{bulk}}(T)$ is the corresponding bulk free energy density.

The full free energy of a d-dimensional critical system in the form of a film with thickness L, area A, and boundary conditions a and b on the two surfaces, at the bulk critical point T_c, has the asymptotic form

$$\begin{aligned} f_{a,b}(T_c,L|d) \cong{}& Lf_{\mathrm{bulk}}(T_c|d) + f^{a}_{\mathrm{surface}}(T_c|d) \\ &+ f^{b}_{\mathrm{surface}}(T_c|d) + L^{-(d-1)}\Delta_{a,b}(d) + \cdots \end{aligned} \tag{1.3}$$

as $A \to \infty$, $L \gg 1$. Here f_{surface} is the surface free energy contribution and $\Delta_{a,b}(d)$ is *the amplitude of the Casimir interaction*. The L dependence of the Casimir term (the last one in Eq. (1.3)) follows from the scale invariance of the free energy and has been derived by Fisher and de Gennes [3]. The amplitude $\Delta_{a,b}(d)$ is *universal*, depending on the bulk universality class and the universality classes of the boundary conditions [4,5].

Equation (1.3) is valid for both *fluid and magnetic systems at criticality*. Prominent examples are, e.g., one-component fluid at the liquid-vapour critical point, the binary fluid at the consolute point, and liquid ^{4}He at the λ transition point [4]. The boundaries influence the system to a depth given by the bulk correlation length $\xi_\infty(T) \sim |T-T_c|^{-\nu}$, where ν is its critical exponent. When $\xi_\infty(T) \ll L$ the Casimir force, as a *fluctuation induced force* between the plates, is negligible. The force becomes long-ranged when $\xi_\infty(T)$ diverges near *and* below the bulk critical point T_c in an $\mathcal{O}(n)$, $n \geq 2$ model system in the absence of an external magnetic field [6,7]. Therefore in statistical-mechanical systems one can turn on and off the Casimir effect merely by changing, e.g., the temperature of the system.

The temperature dependence of the Casimir force for two-dimensional systems has been investigated exactly only

on the example of Ising strips [8]. In $\mathcal{O}(n)$ models for $T > T_c$ the temperature dependence of the force has been considered in [5]. The only example where it is investigated exactly as a function of both the temperature and magnetic field scaling variables is that of the three-dimensional spherical model under periodic boundary conditions [6]. There exact results for the Casimir force between two walls with a finite separation in a $L \times \infty^2$ mean-spherical model have been derived. The force is consistent with an *attraction* of the plates confining the system.

The most of the results available at the moment are for the Casimir amplitudes. They are obtained for $d = 2$ by using conformal-invariance methods for a large class of models [4]. For $d \neq 2$ results for the amplitudes are available via field-theoretical renormalization group theory in $4-\varepsilon$ dimensions [4,5,9], Migdal-Kadanoff real-space renormalization group methods [10], and, relatively recently, by Monte Carlo methods [11]. In addition to the flat geometries recently some results about the Casimir amplitudes between spherical particles in a critical fluid have been derived too [9,12]. For the purposes of experimental verification that type of geometry seems more perspective.

It should be noted that in contrast with the quantum mechanical Casimir effect, that has been tested experimentally with high accuracy [13], the statistical-mechanical Casimir effect lacks so far a satisfactory experimental verification (for comments on the specific difficulties that the experiment stacks with see, e.g. [12]).

In recent years there has been a renewed interest [14, 15] in the theory of zero-temperature quantum phase transitions. In contrast to temperature driven critical phenomena, these phase transitions occur at zero temperature as a function of some non-thermal control parameter, say g, (or a competition between different parameters describing the basic interaction of the system), and the relevant fluctuations are of quantum rather than thermal nature. In the present article we consider a statistical-mechanical Casimir effect when critical quantum fluctuations play an essential role.

It is well known from the theory of critical phenomena that for temperature driven phase transitions quantum effects are unimportant near critical points with $T_c > 0$. It could be expected, however, that at rather low (as compared to characteristic excitations in the system) temperatures, the leading T dependence of all observables is specified by the properties of the zero-temperature critical point, say at g_c. The *dimensional crossover rule* asserts that the critical singularities with respect to g of a d-dimensional quantum system at $T = 0$ and around g_c are *formally* equivalent to those of a classical system with dimensionality $d+z$ (z is the dynamical critical exponent) and critical temperature $T_c > 0$. This makes it possible to investigate low-temperature effects (considering an effective system with d infinite spatial and z finite temporal dimensions) in the framework of the theory of finite-size scaling (FSS). This theory has been applied to explore the low-temperature regime in quantum systems [14,15,16], when the properties of the thermodynamic observables in the finite-temperature quantum critical region have been the main focus of interest.

In this paper a theory of the scaling properties of the free energy and Casimir amplitudes of a quantum spherical model [17] with nearest-neighbor and some special cases of long-range interactions (decreasing at long distances r as $1/r^{d+\sigma}$) is presented. These interactions enter the exact expressions for the free energy only through their Fourier transform wich leading asymptotic is $U(q) \sim q^{\sigma^*}$, where $\sigma^* = min(\sigma, 2)$ [18]. As it was shown for bulk systems by renormalization group arguments $\sigma \geq 2$ corresponds to the case of finite (short) range interactions, i.e. the universality class then does not depend on σ [19]. Values satisfying $0 < \sigma < 2$ correspond to long-range interactions and the critical behaviour depends on σ. On the above reasoning one usually considers the case $\sigma > 2$ as uninteresting for critical effects, even for the finite-size treatments [20]. However recent Monte Carlo results suggest that it might well not be the case at least for continuous Ising model [21]. There Bayong and Diep state that for $d = 2$ the critical exponents does not depend on σ and reach their short-range values for $\sigma \geq 3$. On the basis of that result it seems that for finite-size systems the case $\sigma > 2$ is nontrivial. Since, up to the authors knowledge that is the only example where $\sigma > 2$ is of interest for studying critical properties, here we will consider only the case $0 < \sigma \leq 2$.

The investigation of the Casimir effect in a classical system with long-range interaction possesses some peculiarities in comparison with the short-range system. Due to the long-range character of the interaction there exists a natural attraction between the surfaces bounding the system. One easily can estimate that in the ordered state the L-dependent part of the excess free energy that is raised by the direct inter-particle (spin) interaction is of order of $L^{-\sigma+1}$. In the critical region one still has some effect stemming from that interaction on the background of which develops the fluctuating induced new attraction between the surfaces which is in fact the critical Casimir force. In the definition (1.1) used here, that is the common one when one considers short-range systems, these both effects are superposed simultaneously. Therefore, here, generally speaking, one should expect a crossover from a regime governed by the critical Casimir force (in the sense of a fluctuation induced force; it is of the order of L^{-d}, see Eq. (1.3)) to the one govern by the direct attraction (of the order of $L^{-\sigma}$; note that if $d = \sigma$ they will of the same order being dominating in different temperature regions). An interesting case when forces of similar origin are acting simultaneously is that one of the wetting when the wetting layer is nearly critical and intrudes between two noncritical phases if one takes into account the effect of long-range correlations and that one of the long-range van der Waals forces [22,23].

In quantum systems additional new features will be observed since the "temporal direction" corresponds formally to a short-range type interaction in the corresponding classical analog of the system, i.e. one unavoidable has "anisotropy" in the spectrum of a quantum system with

long-range interactions. The effects depend on which dimension – the temporal or the spatial one – is the finite one. When the spatial dimension is finite we only mention here and will demonstrate in the current article, that an effect similar to that for the classical systems exists. If the finite dimension is the temporal one such effect will not be observed since then the long-range interaction is intersurficial and is not effected directly by the "finite size" of the system.

The plan of the paper is as follows. In Section 2 we define some generalization of Casimir amplitudes in quantum systems and present some hypotheses for the corresponding excess free energies. Then in Section 3 we give a brief review of the model and the basic equations for the free energy and the quantum spherical field in the case of periodic boundary conditions. Since we make use of the ideas of the FSS theory, the bulk system in the low-temperature region is considered as an effective $(d+z)$ dimensional classical system with z finite (temporal) dimensions. This is done to make possible a comparison with other results based on the spherical type approximation, e.g., in the framework of the spherical model and the QNLσM in the limit $n \to \infty$. The scaling forms for the excess free energy, the spherical field equation and the Casimir force are derived for a $\frac{1}{2}\sigma < d < \frac{3}{2}\sigma$ dimensional system with a film geometry in Section 4. In Section 5 we present some results for the Casimir amplitudes in the case of short-range interactions and in some special cases of long-range interactions. The paper closes with concluding remarks given in Section 6.

2 Casimir amplitudes in critical quantum systems

Let us consider a quantum system with a film geometry $L \times \infty^{d-1} \times L_\tau$, where $L_\tau \sim \hbar/(k_B T)$ is the "finite-size" in the temporal (imaginary time) direction and let us suppose that *periodic boundary conditions* are imposed across the finite space dimensionality L (in the remainder we will set $\hbar = k_B = 1$). Let $f(T,g,H;L|d)$ be the free energy density of this system. Then, according to the dimensional crossover rule, the Privman - Fisher hypothesis [24] for finite classical systems and Eq. (1.2) in the quantum case one could state that

$$\frac{1}{L} f^{\text{ex}}(T,g,H;L|d) = (TL_\tau)\, L^{-(d+z)} X^{\text{u}}_{\text{ex}}(x_1,x_2,\rho|d), \tag{2.1}$$

with scaling variables

$$x_1 = L^{1/\nu}\delta g,\ x_2 = hL^{\Delta/\nu} \text{ and } \rho = L^z/L_\tau. \tag{2.2}$$

Here Δ and ν are the usual critical exponents of the bulk model, h is a properly normalized external magnetic field H, $\delta g \sim g - g_c$, and X^{u}_{ex} is the universal scaling function of the excess free energy. According to the definition (1.1) of the Casimir force, one obtains immediately

$$F^d_{\text{Casimir}}(T,g,H;L) = (TL_\tau)\, L^{-(d+z)} X^{\text{u}}_{\text{Casimir}}(x_1,x_2,\rho|d), \tag{2.3}$$

where the *universal* scaling functions of the Casimir force $X^{\text{u}}_{\text{Casimir}}(x_1,x_2,\rho|d)$ is related to the one of the excess free energy $X^{\text{u}}_{\text{ex}} \equiv X^{\text{u}}_{\text{ex}}(x_1,x_2,\rho|d)$ by

$$X^{\text{u}}_{\text{Casimir}}(x_1,x_2,\rho|d) = -(d+z)X^{\text{u}}_{\text{ex}} - \frac{1}{\nu}x_1\frac{\partial X^{\text{u}}_{\text{ex}}}{\partial x_1} + \frac{\Delta}{\nu}x_2\frac{\partial X^{\text{u}}_{\text{ex}}}{\partial x_2} + z\rho\frac{\partial X^{\text{u}}_{\text{ex}}}{\partial \rho}. \tag{2.4}$$

It follows from Eq. (2.3) that depending on the scaling variable ρ one can consider the general case of Casimir amplitudes

$$\Delta^{\text{u}}_{\text{Casimir}}(\rho|d) = X^{\text{u}}_{\text{Casimir}}(0,0,\rho|d)\,. \tag{2.5}$$

The classical amplitudes $\Delta_{a,b}(d)$ (for $(a,b) \equiv$ periodic boundary conditions) introduced by Eq. (1.3) are particular cases of $\Delta^{\text{u}}_{\text{Casimir}}(\rho|d)$ for $\rho = 0$, i.e. $T = 0$ (we remind that our system, according to the dimensional crossover rule, is formaly equivalent to a $d+z$ dimensional classical one).

In addition to the above "usual" excess free energy and Casimir amplitudes, denoted by the superscript "u", one can define, in a full analogy with what it has been done above, "*temporal excess free energy density*" f^{ex}_{t},

$$f^{\text{ex}}_{\text{t}}(T,g,H|d) = f(T,g,H;\infty|d) - f(0,g,H;\infty|d) \tag{2.6}$$

and "*temporal Casimir amplitudes*"

$$f^{\text{ex}}_{\text{t}}(T,g_c,0|d) = TL_\tau^{-d/z}\Delta^{\text{t}}_{\text{Casimir}}(d)\,. \tag{2.7}$$

Whereas the "usual" amplitudes characterize the leading L corrections to the bulk free energy density at the critical point, the "temporal amplitudes" determine the leading temperature-dependent corrections to the ground state energy of an *infinite* system at its quantum critical point g_c.

If in Eq. (2.7) the quantum parameter g is in the vicinity of g_c, then one expects

$$f^{\text{ex}}_{\text{t}}(T,g,H) = TL_\tau^{-d/z}X^{\text{t}}_{\text{ex}}\left(x^t_1,x^t_2|d\right), \tag{2.8}$$

i.e. instead of the amplitude $\Delta^{\text{t}}_{\text{Casimir}}(d)$ one has a scaling function $X^{\text{t}}_{\text{ex}}(x^t_1,x^t_2|d)$ which is the corresponding analog of $X^{\text{u}}_{\text{ex}}(x_1,x_2,\rho|d)$. The scaling variables now are

$$x^t_1 = L^{1/\nu z}\delta g \text{ and } x^t_2 = hL^{\Delta/\nu z}. \tag{2.9}$$

Obviously

$$\Delta^{\text{t}}_{\text{Casimir}}(d) = X^{\text{t}}_{\text{ex}}(0,0|d)\,. \tag{2.10}$$

Let us finally note that if $z = 1$ the temporal excess free energy introduced here coincides, up to a (negative) normalization factor, with the proposed by Neto and Fradkin [25] definition of the non-zero temperature generalization of the C-function of Zamolodchikov.

Now we pass to study the quantities introduced above on the example of one exactly solvable model.

3 The model

The model we consider is described by the Hamiltonian [17]

$$\mathcal{H}=\frac{1}{2}g\sum_{\ell}\mathcal{P}_{\ell}^{2}-\frac{1}{2}\sum_{\ell\ell'}J_{\ell\ell'}\mathcal{S}_{\ell}\mathcal{S}_{\ell'}+\frac{1}{2}\mu\sum_{\ell}\mathcal{S}_{\ell}^{2}-H\sum_{\ell}\mathcal{S}_{\ell}, \tag{3.1}$$

where $\mathcal{S}_\ell$ are spin operators at site ℓ. The operators $\mathcal{P}_\ell$ play the role of "conjugated" momenta (i.e. $[\mathcal{S}_\ell,\mathcal{S}_{\ell'}]=0$, $[\mathcal{P}_\ell,\mathcal{P}_{\ell'}]=0$, and $[\mathcal{P}_\ell,\mathcal{S}_{\ell'}]=i\delta_{\ell\ell'}$, with $\hbar=1$). The coupling constant g measures the strength of the quantum fluctuations (below it will be called quantum parameter), H is an ordering magnetic field, and the spherical field μ is introduced so as to ensure the constraint

$$\sum_{\ell}\left\langle\mathcal{S}_{\ell}^{2}\right\rangle=N. \tag{3.2}$$

Here N is the total number of quantum spins located at sites "ℓ" of a finite hypercubical lattice Λ of size $L_1\times L_2\times\cdots\times L_d=N$ and $\langle\cdots\rangle$ denotes the standard thermodynamic average taken with the Hamiltonian $\mathcal{H}$. In Ref. [17], the equivalence of the model (3.1) and the quantum $\mathcal{O}(n)$ nonlinear sigma model in its large n-limit is shown.

Let us note that in the last few years an increasing interest in the *spherical approximation* (or large n-limit), generating tractable models of quantum critical phenomena, has been observed [17,26,27,28,29,30]. There are different possible ways of quantization of the spherical constraint. In general they lead to *different universality classes* at the quantum critical point [17,26,27,28]. The commutation relations for the operators $\mathcal{S}_\ell$ and $\mathcal{P}_\ell$ together with the kinetic term in the Hamiltonian (3.1) do not describe quantum Heisenberg-Dirac spins but quantum rotors as is pointed out in Ref. [17]. Since the quantum rotors model has been widely exploited in the field of high-temperature superconductivity (see, e.g. [14] and references therein) we hope that the treatment of the model (3.1) presented below might be of some interest to those problems.

For nearest neighbour interaction different low temperature regimes and finite-size scaling properties of the model are investigated in Ref. [29].

The free energy of the model in a finite region Λ under periodic boundary conditions applied across the finite dimensions has the form [30]

$$\beta f_{\Lambda}(\beta,g,H)=\sup_{\mu}\left\{\frac{1}{N}\sum_{q}\ln\left[2\sinh\left(\frac{1}{2}\beta\omega\left(q;\mu\right)\right)\right]-\frac{\mu}{2}\beta-\frac{\beta g}{2\omega^{2}\left(0;\mu\right)}H^{2}\right\}. \tag{3.3}$$

Here the vector q has the components $\left\{\frac{2\pi n_1}{L_1},\cdots,\frac{2\pi n_d}{L_d}\right\}$, $n_j\in\left\{-\frac{L_j-1}{2},\cdots,\frac{L_j-1}{2}\right\}$ for L_j odd integers, and β is the inverse temperature (with $k_B=1$). In (3.3) the spectrum is $\omega^2(q;\mu)=g(\mu+U(q))$ with $U(q)\cong J|q|^\sigma$, $0<\sigma\leq 2$. In the above expressions $U(q)$ is the Fourier transform of the interaction matrix where the energy scale has been fixed so that $U(0)=0$. The supremum in Eq. (3.3) is attained at the solutions of the mean-spherical constraint, Eq. (3.2), that reads

$$1=\frac{t}{N}\sum_{m=-\infty}^{\infty}\sum_{q}\frac{1}{\phi+U(q)/J+b^{2}m^{2}}+\frac{h^{2}}{\phi^{2}}, \tag{3.4}$$

where we have introduced the notations: $b=(2\pi t)/\lambda$, $\lambda=\sqrt{g/J}$ is the normalized quantum parameter, $t=T/J$ - the normalized temperature, $h=H/J$ - the normalized magnetic field, and $\phi=\mu/J$ is the scaled spherical field. Eqs. (3.3) and (3.4) provide the basis for studying the critical behaviour of the model under consideration.

In the thermodynamic limit it has been shown [17] that for $d>\sigma$ the long-range order exists at finite temperatures up to a given critical temperature $t_c(\lambda)$. Here we shall consider the *low-temperature region* for $\frac{1}{2}\sigma<d<\frac{3}{2}\sigma$. We remind that $\frac{1}{2}\sigma$ and $\frac{3}{2}\sigma$ are the lower and the upper critical dimensions, respectively, for the quantum critical point of the considered system.

4 Scaling form of the excess free energy and the Casimir force at low temperatures

For a system with a film geometry $L\times\infty^{d-1}\times L_\tau$ (where $\frac{1}{2}\sigma<d<\frac{3}{2}\sigma$), after taking the limits $L_2\to\infty,\cdots,L_d\to\infty$ in Eq. (3.3) with $L_1=L$, we receive the following expression for the full free energy density (see Appendix A)

$$\begin{aligned}
f(t,\lambda,h;L|d,\sigma)/J=&-\frac{h^{2}}{2\phi}-\frac{\phi}{2}\\
&+\frac{\lambda k_{d}}{2d}x_{D}^{d}\left(x_{D}^{\sigma}+\phi\right)^{\frac{1}{2}}\ {}_{2}F_{1}\left(1,-\frac{1}{2},1+\frac{d}{\sigma},\frac{x_{D}^{\sigma}}{x_{D}^{\sigma}+\phi}\right)\\
&-\frac{\lambda}{4}\frac{\sigma L^{-(d+\frac{\sigma}{2})}}{(4\pi)^{\frac{d}{2}}}\sum_{n=1}^{\infty}\int_{0}^{\infty}dx\,x^{-\frac{\sigma}{4}-\frac{d}{2}-1}\exp\left(-\frac{n^{2}}{4x}\right)\\
&\qquad\times G_{\frac{\sigma}{2},1-\frac{\sigma}{4}}\left(-x^{\sigma/2}L^{\sigma}\phi\right)\\
&-\frac{\lambda}{\sigma}\frac{k_{d}}{\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\phi^{\frac{d}{\sigma}+\frac{1}{2}}\sum_{m=1}^{\infty}\frac{K_{\frac{d}{\sigma}+\frac{1}{2}}\left(m\frac{\lambda}{t}\phi^{\frac{1}{2}}\right)}{\left(m\frac{\lambda}{2t}\phi^{\frac{1}{2}}\right)^{\frac{d}{\sigma}+\frac{1}{2}}}\\
&-\lambda\sqrt{2}\frac{L^{-(d+\frac{\sigma}{2})}}{(2\pi)^{\frac{d+1}{2}}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{0}^{\infty}\frac{dz}{m^{d}z^{\frac{3}{2}}}\mathcal{F}_{\frac{d}{2}-1,\sigma}\left(\frac{z}{m^{\sigma}}\right)\\
&\qquad\times\exp\left[-zL^{\sigma}\phi-\frac{n^{2}}{4zL^{\sigma}}\left(\frac{\lambda}{t}\right)^{2}\right].
\end{aligned} \tag{4.1}$$

Here $k_d^{-1}=\frac{1}{2}(4\pi)^{\frac{d}{2}}\Gamma(d/2)$, x_D is the radius of the sphericalized Brillouin zone,

$$G_{\alpha,\beta}\left(t\right)=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma\left(k+1/2\right)}{\Gamma\left(\alpha k+\beta\right)}\frac{t^{k}}{k!} \tag{4.2}$$

(it was introduced in Ref. [31]),

$$\mathcal{F}_{\nu,\sigma}\left(y\right)=\int_{0}^{\infty}x^{\nu+1}J_{\nu}\left(x\right)\exp\left(-yx^{\sigma}\right)dx \tag{4.3}$$

and $K_\nu(x)$, and $J_\nu(x)$ are the MacDonald and Bessel functions, respectively. The main advantage of the above expression, despite of its complicated form in comparison with Eq. (3.3), is the simplified dependence on the size L which now enters only via the arguments of some functions. This gives us the possibility, as it is explained below, to obtain the scaling functions of the excess free energy and the Casimir force.

In Eq. (4.1) ϕ is the solution of the corresponding spherical field equation that follows by requiring the partial derivative of the r.h.s. of Eq. (4.1) with respect to ϕ to be zero. The bulk free energy $f_{\rm bulk}(t,\lambda,h|d,\sigma)$ results from $f(t,\lambda,h;L|d,\sigma)$ by merely taking the limit $L\to\infty$ in it. Let us denote the solution of the corresponding bulk spherical field equation by ϕ_∞. Then for the excess free energy it is possible to obtain from Eqs. (1.2) and (4.1), in full accordance with Eq. (2.1), the finite size scaling form

$$\frac{1}{L}f^{\rm ex}(t,\lambda,h;L|d,\sigma)=(TL_\tau)\,L^{-(d+z)}X^{\rm u}_{\rm ex}(x_1^u,x_2^u,\rho|d,\sigma), \tag{4.4}$$

with scaling variables

$$x_1^u=L^{1/\nu}\left(\lambda^{-1}-\lambda_c^{-1}\right),\ x_2^u=hL^{\Delta/\nu}\ \text{and}\ \rho=L^z/L_\tau, \tag{4.5}$$

with $L_\tau=\lambda/t$. Here the critical value of $\lambda=\lambda_c$ is

$$\lambda_c^{-1}=\frac{1}{2}(2\pi)^{-d}\int d^dq(U(q)/J)^{-\frac{1}{2}}, \tag{4.6}$$

and $\nu^{-1}=d-\frac{1}{2}\sigma$, $\Delta/\nu=\frac{1}{2}\left(d+\frac{3}{2}\sigma\right)$, and $z=\frac{1}{2}\sigma$ are the critical exponents of the model [17]. In Eq. (4.4) the *universal* scaling function $X^{\rm u}_{\rm ex}(x_1^u,x_2^u,\rho|d,\sigma)$ of the excess free energy has the form

$$\begin{aligned}X^{\rm u}_{\rm ex}(x_1^u,x_2^u,\rho|d,\sigma)&=\frac{1}{2}x_1^u\left(y_\infty-y_0\right)+\frac{1}{2}(x_2^u)^2\left(\frac{1}{y_\infty}-\frac{1}{y_0}\right)\\&-\frac{k_d}{4\sqrt{\pi}\sigma}\Gamma\left(\frac{d}{\sigma}\right)\Gamma\left(-\frac{d}{\sigma}-\frac{1}{2}\right)\left(y_0^{\frac{d}{\sigma}+\frac{1}{2}}-y_\infty^{\frac{d}{\sigma}+\frac{1}{2}}\right)\\&-\frac{k_d}{\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\sum_{m=1}^{\infty}\left[\frac{(2y_0)^{\frac{d}{\sigma}+\frac{1}{2}}K_{\frac{d}{\sigma}+\frac{1}{2}}\left(m\frac{\sqrt{y_0}}{\rho}\right)}{\left(m\frac{\sqrt{y_0}}{\rho}\right)^{\left(\frac{d}{\sigma}+\frac{1}{2}\right)}}\right.\\&\qquad\left.-\frac{(2y_\infty)^{\frac{d}{\sigma}+\frac{1}{2}}K_{\frac{d}{\sigma}+\frac{1}{2}}\left(m\frac{\sqrt{y_\infty}}{\rho}\right)}{\left(m\frac{\sqrt{y_\infty}}{\rho}\right)^{\left(\frac{d}{\sigma}+\frac{1}{2}\right)}}\right]\\&-\frac{1}{4}\frac{\sigma}{(4\pi)^{\frac{d}{2}}}\sum_{n=1}^{\infty}\int_0^\infty x^{-\frac{\sigma}{4}-\frac{d}{2}-1}\exp\left(-\frac{n^2}{4x}\right)\\&\qquad\times G_{\frac{\sigma}{2},1-\frac{\sigma}{4}}\left(-x^{\frac{\sigma}{2}}y_0\right)dx\\&-\frac{\sqrt{2}}{(2\pi)^{\frac{d+1}{2}}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_0^\infty\frac{dz}{m^dz^{\frac{3}{2}}}\mathcal{F}_{\frac{d}{2}-1,\sigma}\left(\frac{z}{m^\sigma}\right)\\&\qquad\times\exp\left[-zy_0-\frac{n^2}{4z\rho^2}\right]\end{aligned} \tag{4.7}$$

(see Appendix A for details of the calculations), where $y_0=\phi L^\sigma$ and $y_\infty=\phi_\infty L^\sigma$.

By direct evaluation of the above expression it is easy to see that below λ_c if the finite system is in ordered phase (then $y_0=y_\infty=0$) $X^{\rm u}_{\rm ex}\sim L^{-\sigma}$, that reflects the dominating contribution of the direct inter-spin long-range interaction in that region [32]. As we see from Eq. (4.4) one observes a crossover from $L^{-(d+z)}$ behavior, where the fluctuation induced interactions dominate, to $L^{-\sigma}$ one where the direct inter-spin interactions become essential.

For the Casimir force one obtains

$$F^{d,\sigma}_{\rm Casimir}(T,\lambda,h;L)=(TL_\tau)\,L^{-(d+z)}X^{\rm u}_{\rm Casimir}(x_1^u,x_2^u,\rho|d,\sigma), \tag{4.8}$$

where the *universal* scaling function of the Casimir force $X^{\rm u}_{\rm Casimir}(x_1^u,x_2^u,\rho|d,\sigma)$ is related to that one of the excess free energy $X^{\rm u}_{\rm ex}\equiv X^{\rm u}_{\rm ex}(x_1^u,x_2^u,\rho|d,\sigma)$ by Eq. (2.4).

The above expressions for the scaling functions of the excess free energy (and the Casimir force) are the most general ones, which gives the possibility of a general analysis including issues as: *i)* the sign of the Casimir force; *ii)* monotonicity of the Casimir force as a function of the temperature; *iii)* the relation of the excess free energy scaling function to the corresponding finite-temperature C-function and its monotonicity properties; *iv)* finite-system generalization of the finite-temperature C-function, etc.

In the present article we will concentrate on evaluation of the Casimir amplitudes for some special cases where one can obtain simple analytical expressions for them.

It is clear that just due to the dimensional crossover rule L plays the same role for the finite system at $t=0$ as L_τ for the corresponding infinite quantum system. Therefore, by a symmetry that obviously arises when $\sigma=2$, one should expect that the behavior of the two types of amplitudes ("normal" and "temporal") will be essentially the same. We will see that explicitly below.

For the model we study here one can show that

$$\begin{aligned}X^{\rm t}_{\rm ex}(x_1^t,x_2^t|d,\sigma)&=\frac{1}{2}x_1^t\left(y_\infty-y_0\right)+\frac{1}{2}(x_2^t)^2\left(\frac{1}{y_\infty}-\frac{1}{y_0}\right)\\&-\frac{k_d}{4\sqrt{\pi}\sigma}\Gamma\left(\frac{d}{\sigma}\right)\Gamma\left(-\frac{d}{\sigma}-\frac{1}{2}\right)\left(y_0^{(d/z+1)/2}-y_\infty^{(d/z+1)/2}\right)\\&-\frac{k_d}{\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)(2y_0)^{\frac{d}{\sigma}+\frac{1}{2}}\sum_{m=1}^{\infty}\frac{K_{\frac{d}{\sigma}+\frac{1}{2}}\left(m\sqrt{y_0}\right)}{\left(m\sqrt{y_0}\right)^{\frac{d}{\sigma}+\frac{1}{2}}}.\end{aligned} \tag{4.9}$$

Here the scaling variables are defined by

$$x_1^t=L_\tau^{1/\nu z}\left(\frac{1}{\lambda}-\frac{1}{\lambda_c}\right),\ x_2^t=hL_\tau^{\Delta/z\nu}, \tag{4.10}$$

$y_0=L_\tau^2\phi_0$ and $y_\infty=L_\tau^2\phi_\infty$. Note that y_0 is the solution of the corresponding spherical filed equation for the nonzero-temperature system, whereas y_∞ is the solution for the zero-temperature (”infinite” in the "temporal" dimension) one. The direct evaluation of (4.9), supposing the finite system in a ordered state, shows that $X^{\rm t}_{\rm ex}$ remains of the same order in that region. This is exactly the behavior to be expected, as we mentioned in the Introduction, despite the long-range nature of the interactions, since the finite "dimension" of the system is now the temporal one.

Now we are ready to investigate in a bit more detail the behavior of the Casimir amplitudes as a function of d, σ and ρ.

5 Evaluation of Casimir amplitudes

In this section we determine the Casimir amplitudes of the model in the case of short-range interactions at $d = 2$ and in the special case $d = \sigma$ of long-range interactions.

5.1 "Usual" Casimir amplitudes

5.1.1 Two-dimensional system with short-range interactions ($d = \sigma = 2$)

In this case essential simplifications in the expression for the Casimir forces can be made. The functions $G_{\alpha,\beta}$ and $\mathcal{F}_{\alpha,\beta}$ used in the general expression of the free energy (4.1) turn into the explicit forms

$$G_{1,1/2}(z) = \frac{1}{\sqrt{\pi}} \exp(z) \tag{5.1}$$

and

$$\mathcal{F}_{\alpha,2}(z) = (2z)^{-\alpha-1} \exp\left(-\frac{1}{4z}\right). \tag{5.2}$$

At the quantum critical point $\lambda = \lambda_c$, $h = 0$ this leads to ($y_\infty = 0$)

$$X^u_{ex}(0,0,\rho|2,2) = -\left(\frac{y_0}{2\pi}\right)^{\frac{3}{2}} {\sum_{m,n}}' \frac{K_{\frac{3}{2}}\left(\sqrt{y_0\left(\frac{n^2}{\rho^2}+m^2\right)}\right)}{\left(\sqrt{y_0\left(\frac{n^2}{\rho^2}+m^2\right)}\right)^{\frac{3}{2}}} - \frac{1}{12\pi} y_0^{3/2}, \tag{5.3}$$

where the primed summation over the integers m and n indicates that the term corresponding to $m = n = 0$ is excluded.

Since it is not clear how to obtain an explicit analytical solution for y_0 in a film geometry at nonzero temperature, the above expression cannot be simplified further, but has to be analyzed numerically (see, e.g., [29] for a numerical analysis of the spherical filed equation). Nevertheless, the above expression can be significantly simplified at zero temperature. Then one shows that the solution y_0 of the spherical field equation for the finite system with a film geometry $L \times \infty \times L_\tau$, at zero temperature (i.e. $1 \ll L \ll \infty$, $L_\tau = \infty$) and at the quantum critical point $\lambda = \lambda_c$, $h = 0$ [29,30] is $y_0 = 4\ln^2\left(\sqrt{5}/2 + 1/2\right)$. Setting this value of y_0 in (5.3), taking into account that $K_{3/2}(x) = \sqrt{\pi/(2x)}\exp(-x)(1+1/x)$, and using the properties of the polylogarithm functions $\mathrm{Li}_p(x)$ [33], we obtain after some algebra that the Casimir amplitude is

$$\Delta^{\mathrm{u}}_{\mathrm{Casimir}}(0|2,2) = -\frac{2\zeta(3)}{5\pi} \approx -0.1530. \tag{5.4}$$

Here $\zeta(x)$ is the Riemann zeta function.

5.1.2 One-dimensional system with long-range interactions ($d = \sigma = 1$)

We can obtain a relatively simple analytical expression from Eq. (4.7) only in the particular case $\sigma = 1$. In this case the functions G and $\mathcal{F}$ become

$$G_{1/2,3/4}(-z) = \frac{\sqrt{z}}{\pi} \exp\left(z^2/2\right) K_{1/4}\left(\frac{z^2}{2}\right) \tag{5.5}$$

and

$$\mathcal{F}_{\nu,1}(y) = \frac{2^{\nu+1}}{\sqrt{\pi}} \Gamma(\nu+3/2) \frac{y}{(1+y^2)^{\nu+3/2}}, \tag{5.6}$$

respectively. In order to obtain explicit results for the amplitudes, numerical evaluations are unavoidable even in the simplest case corresponding to zero temperature. In this case we obtain from Eq. (4.7)

$$X^{\mathrm{u}}_{\mathrm{ex}}(0,0,0|1,1) = -\frac{\sqrt{y_0}}{8\pi^{\frac{3}{2}}} \sum_{\ell=1}^{\infty} \int_0^\infty dx x^{-\frac{3}{2}} \exp\left(\frac{y_0^2 x}{2} - \frac{\ell^2}{4x}\right) \times K_{\frac{1}{4}}\left(\frac{y_0^2 x}{2}\right) - \frac{1}{3\pi} y_0^{3/2}. \tag{5.7}$$

Here y_0 is the solution of the equation for the spherical field which can be obtained by requiring the partial derivative of the r.h.s of (5.7) with respect to y_0 to be zero. Solving the last equation numerically we end up with $y_0 = 0.6248$. After substitution of the solution in Eq. (5.7) we obtain for the Casimir amplitude

$$\Delta^u_{\mathrm{Casimir}}(0|1,1) = -0.3157. \tag{5.8}$$

The result given by Eq. (5.8) shows that the Casimir amplitude in the case $\sigma = 1$ is of larger magnitude than the one in the case of short-range interaction. One can ask whether this is just a coincidence or the Casimir amplitudes are increasing function of σ for a given fixed d.

5.1.3 Relation with the Zamolochikov's C-function

In terms of the critical-point value of an analog of the finite-temperature C-function [25], the result given by Eq. (5.4) can be rewritten in the form

$$\Delta^{\mathrm{u}}_{\mathrm{Casimir}}(0|d,2) = -n^u(d,2)\tilde{c}^u(d,2), \tag{5.9}$$

where in analogy with the short-range interaction case above we define a number $\tilde{c}^u$ via the relation

$$\tilde{c}^u(d,\sigma) = -\Delta^u_{\mathrm{Casimir}}(0|d,\sigma)/n^u(d,\sigma). \tag{5.10}$$

Here, as usual, the normalization factor $n^u(d,\sigma)$ is chosen so that to ensure $\tilde{c}^u = 1$ for the corresponding Gaussian model, i.e.

$$n^u(d,\sigma) = \frac{2^{\sigma/2}\Gamma\left(\frac{d}{2}+\frac{\sigma}{4}\right)\zeta\left(d+\frac{\sigma}{2}\right)}{\pi^{d/2}\left|\Gamma\left(-\frac{\sigma}{4}\right)\right|}. \tag{5.11}$$

For $\sigma = 2$ we immediately obtain $\tilde{c}^u(2,2) = 4/5$, where $n^u(d,2) = \Gamma((d+1)/2)\zeta(d+1)/\pi^{(d+1)/2}$ becomes the normalization factor given in [25]. In that way we reproduce the well known result for $\tilde{c}$ due to Sachdev [33] who considered an example of a three dimensional conformal field theory. This coincidence of the values of $\tilde{c}$ is due to the fact that both models belong to the same universality class. For more details on the behavior of the finite temperature C-function in the case $\sigma = 2, d = 1,2,4$ see, e.g. [34]. The d-dependence of the value $\tilde{c}^u(d,2) \equiv \Delta^{\mathrm{u}}_{\mathrm{Casimir}}(0|d,2)/n^u(d,2)$ has been considered in [35] for d-dimensional $(2 < d < 4)$ conformally invariant field theory. The relation between the C function and the Casimir force for the classical version of the model and short-range interaction has been analyzed in some details for $d = 2$ in [7].

In the particular case of long-range interaction $d = \sigma = 1$, in accordance with Eq. (5.10) and Eq. (5.8) one gets

$$\tilde{c}^u(1,1) = 0.606. \tag{5.12}$$

5.2 "Temporal" Casimir amplitudes

5.2.1 σ-dimensional system $(d = \sigma)$

Let us note that unlike the case of "usual" Casimir amplitudes here it is possible to consider the more general case $d = \sigma$, where $0 < \sigma \leq 2$. From Eq. (4.9) one obtains a general expression for the temporal Casimir amplitudes for a system with geometry $\infty^d \times L_\tau$, at the quantum critical point $\lambda = \lambda_c$, $h = 0$,

$$\begin{aligned}\Delta^{\mathrm{t}}_{\mathrm{Casimir}}(d,\sigma) = &-\frac{k_d}{4\sqrt{\pi}\sigma}\Gamma\left(\frac{d}{\sigma}\right)\Gamma\left(-\frac{2d}{\sigma}-\frac{1}{2}\right)y_0^{\frac{d}{z}+1} \\ &-\frac{k_d}{\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\left(2y_0^2\right)^{\frac{d}{\sigma}+\frac{1}{2}}\sum_{m=1}^{\infty}\frac{K_{\frac{d}{\sigma}+\frac{1}{2}}(my_0)}{(my_0)^{\frac{d}{\sigma}+\frac{1}{2}}}, \end{aligned}\tag{5.13}$$

where the scaling variable y_0 is the solution of the corresponding equation for the spherical field. We notice here that in the particular case $d/\sigma = 1$ Eq. (5.13) simplifies considerably. That is why we are going to investigate namely this case. In a way similar to that explained in the case of short-range interactions, one obtains $(0 < \sigma \leq 2)$

$$\Delta^{\mathrm{t}}_{\mathrm{Casimir}}(\sigma,\sigma) = -\frac{16}{5\sigma}\frac{\zeta(3)}{(4\pi)^{\sigma/2}}\frac{1}{\Gamma(\sigma/2)}. \tag{5.14}$$

Note that the defined "temporal Casimir amplitude" $\Delta^{\mathrm{t}}_{\mathrm{Casimir}}(\sigma,\sigma)$ reduces for $\sigma = 2$ to the "normal" Casimir amplitude $\Delta^{\mathrm{u}}_{\mathrm{Casimir}}(0|2,2)$, given by Eq. (5.4). This reflects the existence of a special symmetry in that case between the "temporal" and the space dimensionalities of the system.

When $\sigma \neq 2$ it is easy to verify that the following general relation

$$\frac{\Delta^{\mathrm{t}}_{\mathrm{Casimir}}(\sigma,\sigma)}{\Delta^{\mathrm{t}}_{\mathrm{Casimir}}(2,2)} = \frac{8\pi}{\sigma(4\pi)^{\sigma/2}\Gamma(\sigma/2)} \tag{5.15}$$

between the temporal amplitudes holds. The r.h.s. of (5.15) is a decreasing function of σ.

5.2.2 Relation with the Zamolochikov's C-function

As it has been already mentioned in Section 2 if $z = 1$ the temporal excess free energy (see Eq. (2.6)) coincides, up to a negative normalization constant, with the nonzero-temperature generalization of the C-function of Zamolodchikov proposed in [25]. For $z \neq 1$ a straightforward generalization of this definition can be proposed at least in the case of long-range power-low decaying interaction

$$C(T,g|d,z) = -T^{-(1+d/z)}\frac{v^{d/z}}{n(d,z)}f^{\mathrm{t}}_{\mathrm{ex}}(T,g), \tag{5.16}$$

where $z = \sigma/2$, the nonuniversal constant v in our notations is $v = TL_\tau$ (see Eq. (2.8)), and the normalization factor is taken to be such that the corresponding Gaussian model with the considered type of interaction will have $C(T,g_c|d,z) \equiv \tilde{c}^t(d,\sigma) = 1$ at its critical point, i.e.

$$n^t(d,\sigma) = \frac{4}{\sigma}\frac{\zeta(1+2d/\sigma)}{(4\pi)^{d/2}}\frac{\Gamma(2d/\sigma)}{\Gamma(d/2)}. \tag{5.17}$$

Let us note that the above choice of the normalization constant $n^t(d,z)$ preserves not only the $\tilde{c}$ value for the Gaussian model, but also the corresponding one for the spherical model if $d = \sigma$. Indeed, in that case from (5.14) and the above definitions one again obtains that

$$\tilde{c}^t(\sigma,\sigma) = 4/5 \tag{5.18}$$

for the spherical model. The result given by Eq. (5.18) is a generalization to the case of long-range interaction of the Sachdev's result.

6 Concluding remarks

In the present article the free energy of a system with a geometry $L \times \infty^{d-1} \times L_\tau$ (where $\frac{1}{2}\sigma < d < \frac{3}{2}\sigma$), is derived (see Eq. (4.1)). For $\sigma = 2$ this general result reduces to the one reported in [30] where only the case of short-range interactions has been considered. The expression (4.4) represents actually the verification of the analog of the Privman-Fisher hypothesis [24] for the finite-size scaling form of the free energy (formulated initially for classical systems) in the case when the quantum fluctuations are essential. Note, that in that case one has a finite space dimension and one additional finite dimension that is proportional to the inverse temperature, which provide different types of critical regimes and Casimir amplitudes. According to the finite-size scaling hypothesis [24,36] one has to expect that the temperature multiplying the universal scaling function will be with exponent $p = 1 + d/z$ [36], where the dynamic-critical exponent z reflects the anisotropic scaling between space and "temporal" ("imaginary-time") directions.

Eqs. (4.7-4.8) present a general expression for the Casimir force in the *quantum* spherical model. In the classical limit $(\lambda = 0)$ for a system with short-range interaction it coincides with the corresponding one derived in [6,7] for the classical spherical model.

In order to derive the Casimir amplitudes in a simple analytical closed form, some particular cases ($d=\sigma$) have been considered:

1) For the short-range case ($\sigma=2$) the corresponding amplitude is given in Eq. (5.4). This amplitude is equal to the "temporal Casimir amplitude" for the $\mathcal{O}(n)$ sigma model in the limit $n\to\infty$ [33]. In the short-range case we have demonstrated *explicitly* that the two models, due to the fact that they belong to the same universality class, indeed possess equal Casimir amplitudes as it is to be expected on the basis of the *hypothesis of universality*.

2) In the long-range case ($d=\sigma\neq 2$), the correction to the ground-state energy of the bulk system due to the nonzero temperature is determined by Eq. (5.14). One observes that in this case $c^t(\sigma,\sigma)=4/5$ does not depend on σ. This could be understood by noting that by changing σ one does not change the exponent in the spectrum that corresponds to the "temporal" (finite-size) dimensionality (see Eq. (3.4)).

3) At zero temperature we evaluated numerically the Casimir amplitude for the particular case $d=\sigma=1$ of the long-range interaction. The result (5.8) shows that the Casimir amplitude is of a larger magnitude than in the case of short range interaction ($\sigma=2$). Furthermore, the universal amplitude $c^u(1,1)$ is no longer σ-independent, because the finite-size part of the spectrum is σ-dependent in this case.

In accordance with the general expectations, all the amplitudes that we have derived are *negative* (see Eqs. (5.3), (5.4), (5.14)) and (5.8).

Finally, let us note that the basic expression (see Eqs. (4.7)) for the scaling function of the excess free energy can be used as a starting point for generalization of some of the existing results on the C-function to the case of long-range interactions. We have suggested in Eq. (5.16) a generalization of the nonzero-temperature C-function, proposed by Neto and Fradkin in [25], to the case of power-law long-range interactions. For the quantum spherical model this definition leads to $\tilde{c}=4/5$ for any $d=\sigma$ which generalizes to long-range interactions the corresponding result for the case of short-range interactions ($d=\sigma=2$) due to Sachdev [33].

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A Mathematical Appendix

First we explain how from Eq. (3.3) one can obtain (4.1) for a system with a geometry $L\times\infty^{d-1}\times L_\tau$. It is easy to see that (3.3) can be rewritten in the form

$$f(t,\lambda,h;L)/J=\frac{t}{N}\sum_{\mathbf{q}}\ln\left\{2\sinh\left[\frac{\lambda}{2t}\sqrt{\phi+\frac{U(\mathbf{q})}{J}}\right]\right\} -\frac{h^2}{2\phi}-\frac{\phi}{2}$$

$$=-\frac{h^2}{2\phi}-\frac{\phi}{2}+A(L)-t\sum_{n=1}^{\infty}U_n(L), \quad \text{(A.1)}$$

where

$$A(L)=\frac{\lambda}{2N}\sum_{\mathbf{q}}\sqrt{\phi+|\mathbf{q}|^\sigma} \quad \text{(A.2)}$$

and

$$U_n(L)=\frac{1}{N}\sum_{\mathbf{q}}\exp\left[-n\frac{\lambda}{t}\sqrt{\phi+|\mathbf{q}|^\sigma}\right]. \quad \text{(A.3)}$$

In order to calculate $A(L)$ we use the identity [31]

$$\sqrt{1+z^\alpha}=\frac{\alpha}{2}\int_0^\infty\left[1-\exp\left(-zt\right)\right]t^{-1-\alpha/2}G_{\alpha,1-\frac{\alpha}{2}}\left(-t^\alpha\right) +1 \quad \text{(A.4)}$$

(see Eq. (4.2) for the definition of $G_{\alpha,\beta}$). Since we are interested in the geometry $L\times\infty^{d-1}\times L_\tau$, one has by standard arguments

$$\frac{1}{N}\sum_{\mathbf{q}}\to\frac{1}{(2\pi)^{d-1}}\int dq^{d-1}\times\frac{1}{L}\sum_{q=-[L-1]/2}^{[L-1]/2}. \quad \text{(A.5)}$$

In the last sum over q, by using the asymptotic formula [37]

$$\frac{1}{L}\sum_q\exp\left[-a\left(\frac{2\pi q}{L}\right)^2\right]\approx\frac{1}{\sqrt{4\pi a}}\left\{\operatorname{erf}\left(\pi\sqrt{a}\right) +2\sum_{l=1}^{\infty}\exp\left[-\frac{l^2L^2}{4a}\right]\right\}, \quad \text{(A.6)}$$

valid for $L>>1$, we obtain, after some algebra,

$$A(L)=A(\infty)+\delta A(L), \quad \text{(A.7)}$$

where

$$A(\infty)=\frac{\lambda}{2}\int_{-\pi}^{\pi}dq_1\ldots\int_{-\pi}^{\pi}dq_d\sqrt{\phi+(q_1^2+\ldots+q_d^2)^{\sigma/2}} \quad \text{(A.8)}$$

and [31]

$$\delta A(L)=-\frac{\lambda}{4}\frac{\sigma}{(4\pi)^{d/2}}\sum_{l=1}^{\infty}\int_0^\infty x^{-\sigma/4-d/2-1}\exp\left(-\frac{l^2L^2}{4x}\right) \times G_{\frac{\sigma}{2},1-\frac{\sigma}{4}}\left(-x^{\sigma/2}\phi\right)dx. \quad \text{(A.9)}$$

Since the only singularities of $A(\infty)$ as a function of ϕ are coming from small q's, it is justified to use a sphericalization of the Brillouin zone, which leads to

$$A(\infty)\simeq\frac{\lambda}{2}k_d\int_0^{x_D}\frac{dx}{x^{1-d}}\sqrt{\phi+x^\sigma} \quad \text{(A.10)}$$

$$\simeq\frac{\lambda k_d}{2d}x_D^d\left(x_D^\sigma+\phi\right)^{1/2} \times{}_2F_1\left(1,-\frac{1}{2},1+\frac{d}{\sigma},\frac{x_D^\sigma}{x_D^\sigma+\phi}\right), \quad \text{(A.11)}$$

where ${}_2F_1$ is the hypergeometric function. Now it is clear how the "first half" of (4.1) can be obtained. Next we turn to evaluation of the term $U_n(L)$. Taking into account (A.5), we rewrite $U_n(L)$ in the form

$$U_n(L) = \frac{L^{-1}}{(2\pi)^{d-1}} \sum_{q=-(L-1)/2}^{(L-1)/2} \int_{-\pi}^{\pi} dq_2 \dots \int_{-\pi}^{\pi} dq_d \times \exp\left[-n\frac{\lambda}{t}\sqrt{\phi + (q_1^2 + \ldots + q_d^2)^{\frac{\sigma}{2}}}\right]. \quad \text{(A.12)}$$

Using the Poisson summation formula

$$\sum_{n=a}^{b} f(n) = \sum_{k=-\infty}^{\infty} \int_a^b dn \exp[i2\pi kn] f(n) + \frac{1}{2}[f(a) + f(b)] \quad \text{(A.13)}$$

we obtain from the above expression

$$U_n(L) = U_n(\infty) + \delta U_n(L), \quad \text{(A.14)}$$

where

$$U_n(\infty) = k_d \int_0^{x_D} dx x^{d-1} \exp\left[-n\frac{\lambda}{t}\sqrt{\phi + x^\sigma}\right] \quad \text{(A.15)}$$

and

$$\delta U_n(L) = \frac{2}{(2\pi)^d} \sum_{l=1}^{\infty} \int_{-\pi}^{\pi} dq_1 \dots \int_{-\pi}^{\pi} dq_d \cos[q_1 l L] \times \exp\left[-n\frac{\lambda}{t}\sqrt{\phi + (q_1^2 + \ldots + q_d^2)^{\sigma/2}}\right] \quad \text{(A.16)}$$

In the low-temperature limit $t << 1$ one can replace in (A.15) x_D by infinity. Then, using the integral representation of $K_\nu(x)$

$$K_\nu(2\sqrt{zt}) = K_{-\nu}(2\sqrt{zt}) = \frac{1}{2}\int_0^\infty \left(\frac{z}{t}\right)^{\frac{\nu}{2}} x^{-\nu-1} \exp\left(-tx - \frac{z}{x}\right) dx \quad \text{(A.17)}$$

we derive

$$U_n(\infty) = \frac{\lambda}{t\sigma}\frac{k_d}{\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\phi^{\frac{d}{\sigma}+\frac{1}{2}} K_{\frac{d}{\sigma}+\frac{1}{2}}\left(n\frac{\lambda}{t}\phi^{\frac{1}{2}}\right) \times \left(n\frac{\lambda}{2t}\phi^{\frac{1}{2}}\right)^{-\left(\frac{d}{\sigma}+\frac{1}{2}\right)}. \quad \text{(A.18)}$$

We are left to deal now only with $\delta U_n(L)$. Sphericalizing the Brillouin zone in (A.16), performing the integrations, by using the integral representation for the Bessel function $J_\nu(z)$

$$\int_{-a}^{a} \left(a^2 - x^2\right)^{\beta-1} \exp[i\lambda x]\, dx = \sqrt{\pi}\Gamma(\beta)\left(\frac{2a}{\lambda}\right)^{\beta-1/2} J_{\beta-1/2}(a\lambda) \quad \text{(A.19)}$$

(Re $\beta > 0$) we get

$$\delta U_n(L) = \frac{2L^{-d/2+1}}{(2\pi)^{d/2}} \sum_{l=1}^{\infty} \int_0^{x_D} \frac{x^{d/2}}{l^{d/2-1}} J_{d/2-1}(lLx) \times \exp\left[-n\frac{\lambda}{t}\sqrt{\phi + x^\sigma}\right]. \quad \text{(A.20)}$$

In the low-temperature limit the upper limit of integration in the above expressions can be replaced by infinity. From (A.18) and (A.20) one obtains the last two terms in Eq. (4.1).

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